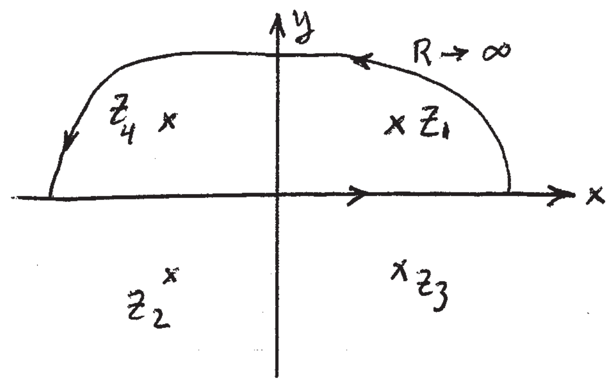
Problem 21) $f(z) = z^4 - 2\cos(2\theta) z^2 + 1 = 0 \Rightarrow$

$$z^2 = \cos(2\theta) \pm \sqrt{\cos^2 2\theta - 1} = \cos(2\theta) \pm i \sin(2\theta) = e^{\pm i 2\theta}$$

$\Rightarrow z = \pm e^{\pm i\theta}$. Therefore $z_1 = e^{i\theta}$, $z_2 = -e^{i\theta}$, $z_3 = e^{-i\theta}$, $z_4 = -e^{-i\theta}$.



We use a semi-circular contour in the upper-half-plane in the limit when $R \to \infty$. (We could as well use a similar contour in the lower-half-plane.) Since the integrand goes to zero (when $R \to \infty$) on the semi-circular part of the contour sufficiently rapidly (in (a) the integrand goes to zero as z^{-4}; in (b) it goes to zero as z^{-2}) we'll have:

a) $$\int_{-\infty}^{\infty} \frac{dx}{x^4 - 2\cos(2\theta) x^2 + 1} = \oint_C \frac{dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$= 2\pi i \,(\text{Residue at } z_1 + \text{Residue at } z_4) = \frac{2\pi i}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{2\pi i}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}$$

$$= \frac{2\pi i}{(e^{i\theta}+e^{i\theta})(e^{i\theta}-e^{-i\theta})(e^{i\theta}+e^{-i\theta})} + \frac{2\pi i}{(-e^{-i\theta}-e^{i\theta})(-e^{-i\theta}+e^{i\theta})(-e^{-i\theta}-e^{i\theta})}$$

$$= \frac{2\pi i}{2e^{i\theta}(2i\sin\theta)(2\cos\theta)} + \frac{2\pi i}{(-2\cos\theta)(2i\sin\theta)(-2e^{-i\theta})}$$

$$= \frac{\pi}{4\sin\theta\cos\theta}\left(e^{-i\theta} + e^{i\theta}\right) = \frac{\pi}{2\sin\theta}.$$

b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4 - 2\cos(2\theta)x^2 + 1} = \oint_C \frac{z^2\,dz}{z^4 - 2\cos(2\theta)z^2 + 1} =$

$$\oint \frac{z^2\,dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 2\pi i\,(\text{Residue at } z_1 + \text{Residue at } z_4)$$

$$= \frac{2\pi i\, z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{2\pi i\, z_4^2}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}$$

$$= \frac{2\pi i\, e^{2i\theta}}{8i \sin\theta \cos\theta\, e^{i\theta}} + \frac{2\pi i\, e^{-2i\theta}}{8i \sin\theta\cos\theta\, e^{-i\theta}}$$

$$= \frac{\pi e^{i\theta}}{4\sin\theta\cos\theta} + \frac{\pi e^{-i\theta}}{4\sin\theta\cos\theta} = \frac{2\pi \cancel{\cos\theta}}{4\sin\theta\cancel{\cos\theta}} = \frac{\pi}{2\sin\theta}.$$

If the semi-circular contour in the lower-half-plane is chosen, the direction of travel around the contour will be clockwise, in which case the small circles going around the poles at z_2 and z_3 will be going counterclockwise. The net result is that the residues must be multiplied by a minus sign. (The final result of the integral, however, will be the same.)